Section «Mathematics and mechanics»

The limit set of the Henstock-Kurzweil integral of a vector-valued function

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We introduce the notion of the limit set of the Henstock-Kurzweil integral $I_{H-K}(f)$ of a function $f : [0,1] \to X$, where X is a Banach space, and study its properties. Similar notion of a limit set and properties for Riemann integral is studied in [2]. Henstock -Kurzweil integral generalizes Riemann and Lebesgue integrals and is used for integration of highly oscillatory functions which occur in quantum theory and nonlinear analysis (see [3]). Also one may consider integral and differential equations using Henstock-Kurzweil integral (see [1]).

Theorem 1 *Let X be a Banach space and for a function $f : [0,1] \to X$ its image $f([0,1])$ is relatively compact in X. Then f is integrable if and only if its limit set $I_{H-K}(f)$ consists exactly of one point and under this assumption its integral is exactly this point.*

However it appears that a one-point limit set does not guarantee the existence of the integral. Our main result is a construction of an example that illustrates this

Theorem 2 *There exists a function $f : [0,1] \to \ell_1[0,1]$ such that its limit set $I_{H-K}(f)$ consists exactly of one point, but this function is not Henstock-Kurzweil integrable.*

In addition we establish general properties of limit sets of the Henstock-Kurzweil integral

Theorem 3 *Let $f : [0,1] \to X$, where X is a Banach space, then the limit set of f enjoys the following properties:*

- *if T is a continuous linear map and $x \in I_{H-K}(f)$, then $Tx \in I_{H-K}(Tf)$;*
- *if $f([0,1])$ is relatively compact in X, then $I_{H-K}(f)$ is convex;*
- *let $g : [0,1] \to X$ and image of f or of g is relatively compact in X, then*

$$I_{H-K}(f+g) = I_{H-K}(f) + I_{H-K}(g).$$

References

1. S. Heikkila, Differential and integral equations with Henstock-Kurzweil integrable functions// J. Math. Anal. Appl. 379. 2011. p. 171-179.

2. M.I. Kadets, V.M. Kadets, Series in Banach spaces. Conditional and unconditional convergence / Birkhauser, Basel - Boston - Berlin, 1997.

3. Douglas S. Kurtz, Charles W. Swartz. Series in real analysis -volume 9: theories of integration. The integrals of Riemann, Lebesgue, Henstock-Kurzweil and McShane, World scientific, 2004.