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Singularities of integrable systems: a criterion for non-degeneracy and a generalization monodromy, with an application to the Manakov top

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A *(completely) integrable Hamiltonian system* $(M, \omega, h_1, \ldots, h_n)$ is a symplectic $2n$-manifold (M, ω) with functionally independent commuting functions $h_1, \ldots, h_n : M \to \mathbb{R}$ called *integrals*. The *momentum map* $\mathcal{F} : M \to \mathbb{R}^n$ given by $\mathcal{F}(x) := (h_1(x), \ldots, h_n(x))$. The Hamiltonian vector field of a function g on M is denoted by sgrad g. A point $x \in M$ is called a *singular (critical) point of rank* r, $0 \le r$, if $\mathrm{d}\mathcal{F}|_x = r$. For such points, there is a natural notion of non-degeneracy [1], [2]. We recall this definition for zero-rank critical points below.

In general, non-degenerate singularities are important because they are generic and because the structure of integrable systems in their neighborhood is well understood, see survey [3]. We present a geometric criterion for non-degeneracy of a singularities of integrable Hamiltonian systems, see Theorem 1 below.

The criterion is applied to prove non-degeneracy of the saddle-saddle singularity in the Manakov top system [4]. We use the proved non-degeneracy and Fomenko theory [2] to obtain explicit semilocal description of the singularity. This description allows to detect a phenomenon which appears in the Manakov top. It naturally generalizes Hamiltonian monodromy introduced by Duistermaat [5] and bidromy introduced recently by Sadovskii and Zhilinskii [6]. We call it *partial monodromy.* Sinitsyn and Zhilinskii [4] showed that no Hamiltonian monodromy in any previously known sence appears in the Manakov top and encouraged further study of the system; our results develop the topic.

Definition 1. Let $(M, \omega, h_1, \ldots, h_n)$ be an integrable Hamiltonian system. A zero-rank singular point $P \in M$ is called *non-degenerate* if the commutative subalgebra K of $sp(2n, \mathbb{R})$ generated by linear parts of Hamiltonian vector fields sgrad $h_1, \ldots,$ sgrad h_n at point P is a Cartan subalgebra of $sp(2n, \mathbb{R})$.

Theorem 1. *Consider a completely integrable Hamiltonian system* $(M, \omega, h_1, \ldots, h_n)$. *Let* $\mathcal{F} : M \to \mathbb{R}^n$ *be the momentum map and* $P \in M$ *be a zero-rank singular point of the system. Denote by* K *the set of all singular points of rank 1 in a neighborhood of* P. *If the following conditions hold, then* P *is non-degenerate:*
(a) There exists a non-degenerate linear combination of forms $\{\mathrm{d}^2 h_i|_P\}_{i=1}^n$.
(b) The image $\mathcal{F}(K)$ *contains* n *smooth curves* $\gamma_1, \ldots, \gamma_n$, *each curve having* P *as its end point or its inner point (examples for* $n = 2$ *are found on the figure). The vectors tangent to* $\gamma_1, \ldots, \gamma_n$ *at* $\mathcal{F}(P)$ *are independent in* $\mathbb{R}^n$.
(c) K *is a smooth submanifold of* M *or, at least,* $K \cup \{P\}$ *coincides with the closure of the set of all points* $x \in K$ *having a neighborhood* $V(x) \subset M$ *for which* $K \cap V(x)$ *is a smooth submanifold of* M.

Remark. Notice that condition (c) is very weak. For example, it automatically holds if the integrals h_i are polynomials (in a suitable system of local coordinates at point P) because in this case each D_i is given by a system of algebraic equations.

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Иллюстрации

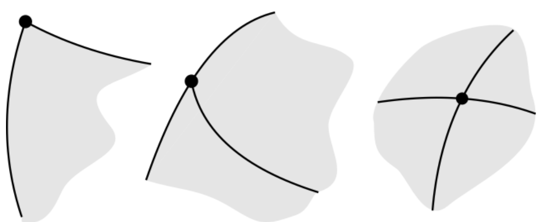

Рис. 1: Bifurcation diagrams satisfying condition (b) of Theorem 1.